eigenproblem of (block-) circulant matrices

Let $A = \begin{pmatrix} a_0 & a_{N-1} & \cdots & a_1 \\ a_1 & a_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{N-1} \\ a_{N-1} & a_{N-2} & \cdots & a_0 \end{pmatrix}$ be a circular matrix.

What are the eigenvalues and eigenvectors of A ?

Let $Q = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & 1 & & \\ \vdots & & \ddots & \\ 0 & \cdots & 1 & 0 \end{pmatrix}$ Q denotes the shuffle by 1 step.
Q^k - - - - - - - - k steps.

Notice that $A = a_0 I + a_1 Q + \cdots\cdots + a_{N-1} Q^{N-1}$.

Let $U(k,l) = \frac{1}{\sqrt{N}} e^{-2\pi j \frac{kl}{N}}$ (DFT matrix)$\cdot\sqrt{N}$

Fix k. Let $\xi = e^{-\frac{2\pi j}{N}}$, $p(x) = a_0 + a_1 x + \cdots + a_{N-1} x^{N-1}$.

Then $Q^m \overrightarrow{\left(e^{-2\pi j \frac{kl}{N}}\right)} = Q^m \begin{pmatrix} e^{-2\pi j \frac{k\cdot 0}{N}} \\ e^{-2\pi j \frac{k\cdot 1}{N}} \\ \vdots \\ e^{-2\pi j \frac{k(N-1)}{N}} \end{pmatrix} = \underbrace{e^{-2\pi j \frac{k \bmod_N(m)}{N}}}_{= \xi^{k \bmod_N(m)}} \overrightarrow{\left(e^{-2\pi j \frac{kl}{N}}\right)}$

So, $U Q^m U^{-1} = \mathrm{diag}\left(1, \xi^{\bmod_N(m)}, \xi^{2 \bmod_N(m)}, \cdots\cdots, \xi^{(n-1)\bmod_N(m)}\right)$

$\Rightarrow U A U^{-1} = U P(Q) U^{-1}$

$= \mathrm{diag}\left(p(1), p(\xi), \cdots\cdots, p(\xi^{N-1})\right) =: \tilde{A}$

$\Rightarrow A = U^{-1} \underbrace{\tilde{A}}_{\text{diagonal}} U$

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So, the eigenvalues of A are $P(1), P(\xi), \cdots\cdots, P(\xi^{N-1})$.

The corresponding eigenvectors are columns of $(\sqrt{N}\,U)^*$,

which are $\vec{\omega}_k = \begin{pmatrix} 1 \\ e^{\frac{2\pi j}{N}k} \\ \vdots \\ e^{\frac{2\pi j}{N}(N-1)k} \end{pmatrix}$, $k = 0, \cdots, N-1$.

A direct consequence is that all circulant matrices share same eigenvectors $\vec{\omega}_k$, $k = 0, \cdots, N-1$.

Next, we move on to block circulant matrices.

Let $H = \begin{pmatrix} H_0 & H_{N-1} & \cdots & H_1 \\ H_1 & H_0 & \ddots & H_2 \\ \vdots & \vdots & \ddots & \vdots \\ H_{N-1} & H_{N-2} & \cdots & H_0 \end{pmatrix}$ be block circulant.

(each H_i is circulant).

Similarly, let $Q = \begin{pmatrix} 0 & & \cdots & & 1 \\ 1 & 0 & & & \\ \vdots & 1 & \ddots & & \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & & 1 & 0 \end{pmatrix}$

Then, $H = I \otimes H_0 + Q \otimes H_1 + Q^2 \otimes H_2 + \cdots\cdots + Q^{N-1} \otimes H_{N-1}$

One important property of Kronecker product:

Let A, B, C, D be matrices s.t. AC and BD are well-defined.

Then, $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

(can be proved by direct computation) or properties of tensor product.

One observation (HW3 Q3)

Let $W_N(n,k) = \frac{1}{\sqrt{N}} e^{2\pi j \frac{nk}{N}}$.

W_N is unitary. $W_N^{-1}(n,k) = W_N^*(n,k) = \overline{W_N}(n,k) = \frac{1}{\sqrt{N}} e^{-2\pi j \frac{nk}{N}}$

Let $W = W_N \otimes W_N$. Then, $W^{-1} = \overline{W_N} \otimes \overline{W_N}$.

Then, $W(Q^m \otimes H_m) W^{-1}$

$$= (W_N \otimes W_N)(Q^m \otimes H_m)(\overline{W_N} \otimes \overline{W_N})$$

$$= (W_N \otimes W_N)\Big((Q^m \overline{W_N}) \otimes (H_m \overline{W_N})\Big)$$

$$= (W_N Q^m W_N^{-1}) \otimes (W_N H_m W_N^{-1})$$

By previous arguments,

$$W_N Q^m W_N^{-1} = \operatorname{diag}\left(1, \xi^{\operatorname{mod}_N(m)}, \xi^{2\operatorname{mod}_N(m)}, \ldots\ldots, \xi^{(n-1)\operatorname{mod}_N(m)}\right)$$

$$W_N H_m W_N^{-1} = \operatorname{diag}\left(P_m(1), P_m(\xi), \ldots\ldots, P_m(\xi^{N-1})\right) =: \tilde{H}_m$$

So, $W H W^{-1} = W\left(I \otimes H_0 + Q \otimes H_1 + \cdots\cdots + Q^{N-1} \otimes H_{N-1}\right) W^{-1}$

$$= \underbrace{\tilde{H}}_{\text{diagonal}}$$

Remark: In the lecture notes, you see the diagonalization of two matrices D and L. The formula follows from the above and Q3(b) in HW3.

How can we accelerate computation involving circular matrices using FFT?

Let A be a circular matrix.

$$A = \begin{pmatrix} a_0 & a_{N-1} & \cdots & a_1 \\ a_1 & a_0 & \ddots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1} & a_{N-2} & \cdots & a_0 \end{pmatrix}$$

Let x be a vector,

$$x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix}$$

Then, $$A^T x = \begin{pmatrix} a_0 x_0 + a_1 x_1 + \cdots + a_{N-1} x_{N-1} \\ a_{N-1} x_0 + a_0 x_1 + \cdots + a_{N-2} x_{N-1} \\ a_{N-2} x_0 + a_{N-1} x_1 + \cdots + a_{N-3} x_{N-1} \\ \vdots \\ a_1 x_0 + a_2 x_1 + \cdots + a_0 x_{N-1} \end{pmatrix}$$

$\Rightarrow A^T x = a * x$, where a is a vector.

To compute $A^T x$, computation complexity is $O(N^2)$.

$A^T x = a * x$, $\mathrm{DFT}(a * x) = \mathrm{DFT}(a) \odot \mathrm{DFT}(x)$

$$a * x = \underbrace{\mathrm{iDFT}}_{O(N\log N)} \left[\underbrace{\mathrm{DFT}}_{\substack{O(N\log N) \\ \text{using FFT}}}(a) \odot \underbrace{\mathrm{DFT}(x)}_{O(N\log N)} \right]$$

$\Rightarrow$ total cost: $O(N\log N)$